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Math 415 linear algebra

Matrix

Matrix Algebra:

* Sum:
* Multiply with Real Number:

Sum of matrices and Multiply of matrices and real number are called **Linear Operation of Matrices**

* Multiply with Matrix: If A is a matrix, B is a matrix,

and C is a matrix.

* Transpose of Matrix:

Inverse: If , is invertible and B is A’s **Inverse Matrix,**

**Elementary Operation and Linear Equation System**

Elementary Row Operations:

Inverse Row Operations

Equivalent:

* if then
* if then

A matrix is said to be in **row echelon form**

**(i)**If the first nonzero entry in each nonzero row is 1.

**(ii)**If row *k* does not consist entirely of zeroes, the number of leading zero entries in row *k* + 1 is greater than the number of leading zero entries in row *k*.

**(iii)**If there are rows whose entries are all zero, they are below the rows having nonzero entries.

Theorem 1: If A and B are both matrices, then:

* A row equivalent to B Order-m invertible matrix P that
* A column equivalent to B Order-m invertible matrix Q that
* A equivalent to B Order-m invertible matrix P and Q that

Reduced Row Echelon Form:

1. The matrix is in row echelon form.
2. The first nonzero entry in each row is the only nonzero entry in its column.

Rank(A): 矩阵A最高阶非零子式的阶数

* If then

Theorem 3: For n-variable linear equation

1. 有唯一解

Theorem 4: For 有非零解

Theorem 7: If , then

Linear Space

Linear Space: V is a non-empty set, is Real Number field. V with two operations + and is called a **linear space** if it has the following properties:

1. V对
2. (零元)

满足(ii)-(viii)的加法和数乘运算称为线性运算(Linear operation).

Dimension: In linear space V, if there exists n elements , that

1. linear irrelevant;
2. Any arbitrary element can be linear expressed by

Then, is called a **basis** of linear space V, n is called linear space V’s **dimension**.

Linear Space V with dimension n is called n-dimension linear space, note as .

有序数(coordinate).

Basis Transform and Coordinate Transform:

Suppose and are two basis on linear space

Matrix P is the **Transition Matrix** from basis to basis

Theorem 1: has coordinate under basis, it’s coordinate under basis is

or

计算过渡矩阵P的方法：

Linear Transformation

Mapping: Suppose are n-dimension and m-dimension linear space, T is a map from to if:

1. For arbitrary ,
2. For arbitrary

Then, T is linear mapping from to , or **Linear Transform.**

**Specially, if T is a linear transform from to itself, it’s called a Linear Transform within .**

Properties:

* ；
* If , ；
* If are linear relevant, are linear relevant as well；
* is a linear space, called **Image space；.**
* **Kernel**: is also a linear space.

T

Im(T)

Ker(T)

0

Suppose T is a linear transformation within linear space , and pick a basis . If we use matrix to express the transformation as following

Is called the matrix of linear transformation T under basis .

The number of dimension of image space is called the **rank of linear transformation T.**

Geometry:

* Scaling by k times
* Orthogonal Projections
* Reflection
* Rotation
* Shear
  + Vertical
* Horizontal

Subspace: A subset W of vector space is called a subspace of W if it has the following properties:

* W contains the zero vector
* W is closed under linear combinations

Theorem: Image and Kernel are subspaces

Vector Group

Consider vectors in ,

* We say that a vector is **redundant** if it is a linear combination of the proceeding vectors
* Vectors **is linearly independent** if none of them is redundant
* form a **basis** of a subspace V if they span V and are linearly independent

Theorem: Consider in **.** If is nonzero, and if each of the vectors has a nonzero entry in a component where all the preceding vectors have a 0, then the vectors are linearly independent.

Consider vectors in . An equation of form

There’s always the trivial relation , while nontrivial relations are not guaranteed to exist.

Theorem: The vectors are linearly dependent if and only if there are nontrivial relations among them.

Theorem: The vectors in the kernel of an matrix A correspond to the linear relations among the column vectors of A, the following arguments are equivalent:

* vectors are linearly independent
* None of the vectors is redundant
* None of the vectors is a linear combination of the other vectors

Theorem:

(nullity)

Similar Matrix & Quadratic Form

Inner Product:

Property:

* if , ; else,
* Schwarz Inequality:

Length:

when

Property:

Theorem: If n-dimension vector is a vector group that any two vectors are orthogonal with each other, then are linearly independent.

Orthonormal Basis: Let n-dimension vector is a basis of vector space V(), if are inter-orthogonal and are all unit vectors, then is a **Orthonormal Basis** of V.

And coordinate under **Orthonormal Basis** is determined by:

Gram-Schmidt Process:

Orthogonal Matrix: If A is a **orthogonal matrix**

Property:

* is also an orthogonal matrix；
* |A| = 1 or -1；
* If A and B are both orthogonal matrix, AB is an orthogonal as well.

QR Factorization: Consider an matrix *M* with linearly independent columns *.* Then there exists an *n x m* matrix *Q* whose columns are orthonormal and an upper triangular matrix *R* with positive diagonal entries such that

This representation is unique.

Orthogonal Transformation:

If P is an orthogonal matrix, then linear transformation is called orthogonal transformation.

Theorem: Orthogonal Transformation preserves orthogonality.

Isomorphism: An invertible linear transformation T is called an **isomorphism**. We say that the linear space V is isomorphic to the linear space W if there exists an isomorphism T from V to W.

Theorem: Coordinate Transformations

are isomorphism.

Property:

* A linear transformation *T* from *V* to *W* is an isomorphism if (and only if) ker(T) = {0} and im*(T) = W.*
* b. If *V* is isomorphic to *W,* then dim(*V)* = dim(W).
* Suppose *T* is a linear transformation from *V* to *W* with ker(T) = {0}. If dim(V) = dim(W), then *T* is an isomorphism.
* Suppose *T* is a linear transformation from *V* to *W* with im*(T)* = W. If dim(V) = dim(W), then *T* is an isomorphism.

T

A

B

S

S

Least Squares solution: Consider a linear system

Where A is an matrix. A vector in is called **least-squares solution** of this system if for all in .

Theorem 1: The least-squares solution of the system

are the exact solutions of the (consistent) system

The system is called the normal equation of .

Theorem 2: If , then the linear system

has the unique least-squares solution:

is the matrix of the orthogonal projection onto V.

Inner Space: A linear space endowed with an inner product is called an **inner product space**.

Property:

* <f, g> = <g, f>
* <f + h, g> = <f, g> + <h, g>
* <cf, g> = c<f, g>
* <f, f> > 0

Trace:

Orthogonal Projection: If is an orthogonal basis of a subspace W of an inner product space V, then

Fourier Analysis:

1. Euler Identities

Theorem: Let be the space of all trigonometric polynomials of order < *n,* with the inner product

then the functions

form an orthogonal basis of .

If *f* is a piecewise continuous function defined on the interval , then its best approximation in is

where

Determinants

Determinant:

Property:

1. ;
2. If there are two totally identical rows or columns, D = 0;
3. Det (AB) = Det (A) Det (B)
4. If A is similar with B, Det (A)=Det (B)

Minor: For an matrix A, let be the matrix obtained by omitting the ith row and the yth column of A. The determinant of thematrix is called a minor of A.

Laplace expansion:

Geometry Interpretation:

* An orthogonal matrix A with det A = 1 is called a *rotation matrix,* and the linear transformation is called a *rotation.*

Cramer’s Rule: Consider the linear system

Where A is an invertible matrix. The components of the solution vector are

where is the matrix obtained by replacing the ith column of A by .

Eigenvalue & Eigenvector

Let ***A*** a n-dimension matrix, if real number and n-dimension nonzero column vector ***x*** that

then is Matrix A’s **eigenvalue**, and **x** is **A**’s **eigenvector** correspond to .

has nonzero solution**,**

or

which has n solutions.

Property:

If is A’s eigenvalue,

Theorem: Suppose are A’s m eigenvalues, and are correspond eigenvectors. If none of any two value in are equal to each other, is linearly independent.

Similar matrix: Let A, B to be n-order square matrix, if there exist an invertible matrix P that

we say that B is A’s **similar matrix,** and the performance of operation is called to perform a **similar transformation**, P is called to be A’s **similar transformation matrix** that transform A into B.

Theorem: If A is similar with B, A and B’s Eigen-polynomials are the same, as well as the eigenvalue.

Corollary: If n-order matrix A and diagonal matrix

are similar, are A’s n eigenvalues.

Diagonalization: If and only if A has n linearly independent eigenvectors, there exists matrices P that

where P is formed by the orthonormalized eigenvectors.

And,

Stability: Consider a dynamic system

We say that is am (asymptotically) stable equilibrium for this system if

for all its trajectories.

Theorem: Consider a dynamical system

The zero state is asymptotically stable if (and only if) the modulus of all the complex eigenvalues of A is less than 1.



Theorem: Consider the dynamical system *,* where *A* is a real matrix with eigenvalues

Let be an eigenvector of *A* with eigenvalue *.*

Then

Note that is the coordinate vector of with respect to basis .

If *r* = 1, then the points are located on an ellipse; trajectory spirals outward; and if *r* is less than 1, then the trajectory spirals inward, approaching the origin.

Quadratic Form

Theorem: A matrix *A* is *orthogonally diagonalizable* (i.e., there exists an orthogonal *S* such that is diagonal) if and only if *A* is *symmetric* (i.e., ).

Theorem: A symmetric matrix *A* has *n* real eigenvalues if they are counted with their algebraic multiplicities.

Quadratic Form: A function from to is called a *quadratic form* if it is a linear combination of functions of the form (where i and j may be equal).A quadratic form can be written as

for a unique symmetric matrix A, called the matrix of q.

Theorem: Consider a quadratic form , where A is a symmetric matrix. Let be an orthonormal eigenbasis for A, with associated eigenvalues . Then

where the are the coordinates of with respect to .

Definiteness: Consider a quadratic form , where A is a symmetric matrix. We say that

* A is **positive definite** if , and we call A **positive semidefinite** if , for all .
* A is **negative definite** if , and we call A **negative semidefinite** if , for all .
* Else, A is called **indefinite**.

Theorem: A symmetric matrix *A* is *positive definite* if (and only if) all of its eigenvalues are positive. The matrix *A* is *positive semidefinite* if (and only if) all of its eigenvalues are positive or zero.

Theorem: Consider a symmetric matrix *A.* For *m* = 1........*n,* let *)* be the matrix obtained by omitting all rows and columns of *A* past the wth. These matrices *A (m)* are called the ***principal submatrices***of *A.*

The matrix A is positive definite if (and only if) , for all m = 1, ..., n.

Congruence: Suppose A and B are matrices with order n, if there exists an invertible matrix C, that , we say that A is **congruent** to B.

Sylvester's law of inertia: Suppose there is Quadratic Form with rank r, and two invertible transformation

to make

and

then and has the same number of positive number.

Theorem: n-variable quadratic form is positive definite if and only if all the index of its standard form are positive.

Hurwitz theorem：

* Symmetric matrix A is positive definite
* Symmetric matrix A is negative definite

Principle axes: Consider a quadratic form *,* where is *a* symmetric matrix with *n* distinct eigenvalues. Then the eigenspaces of *A* are called the *principal axes* of *q.*